2008

MATHEMATICS CURRICULUM

CORE CONCEPTS, SKILLS, AND PROCEDURES

BY GRADE LEVEL

2008

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Center for Teaching/Learning Of Mathematics

Mission:

The mission of CT/LM is to improve mathematics instruction for all—children as well as adults.
The goal of the Center is to provide access to more meaningful mathematics for more learners in effective ways.
To help students, both gifted and learning disabled, realize their potential in mathematics.

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PHILOSOPHY AND GUIDING PRINCIPLES

The following pages describe the key ideas and principles that I believe should guide a mathematics curriculum and mathematics instruction in our schools. The curriculum should emphasize competence, fluency, and proficiency rather than exposure. By the eighth grade, children should have mastered: arithmetic facts and procedures, fractions, integers, and algebra.

Our current standards are low, and there is lack of curricular alignment from grade to grade. Children can acquire considerable knowledge of numbers and operations in the earlier grades that will serve as foundations of higher mathematics. Young children are developmentally ready to acquire strong numeracy skills in the early grades so that they are prepared for algebra in eighth grade. Unfortunately, not all children in eighth grade have access to algebra.

Our mathematics curriculum and instruction should make it clear to children that effort in mathematics is critical. Children from high achieving countries believe that effort is responsible for success. In contrast, American children believe that it depends on ability. A change from a focus on ability to a focus on effort will increase engagement and persistence. Effective teachers have always known that children's beliefs about learning are related to their mathematics performance. The impact on students' mathematics learning is compounded if students have a series of these effective teachers. Effective teachers know their content well. They understand their children's needs. And they employ strategies and communication skills to connect meaningful content with learners.

Even the best teachers need textbooks to follow the curriculum. But textbooks do not teach. Teachers teach. Unlike mathematics texts from other countries, American textbooks are too long, sometimes incorrect, and not focused. Our teachers and children deserve better. However, effective teachers are capable of supplementing textbooks.

Effective teachers and improved textbooks advance the teaching of mathematics for all. But we also need to meet the needs of mathematically gifted students. Providing talented students with excellent teaching, extra resources, and accelerated programs will give us an edge in scientific innovations.

We must not waste times on issues that divide mathematics educators: *learning basic skills vs. conceptualization*; *use of calculators in elementary school*; *concrete models* or *symbolic manipulators; constructivism vs. formal teaching*; *student-centered vs. teacher-led classrooms.*

In order to understand and communicate mathematical ideas, a student must master its language, concepts, and procedures with fluency and confidence. An understanding of mathematical operations, fluent execution of procedures, and fast recall of arithmetic facts jointly support skilled problem solving and recognition of patterns. To identify patterns, children need to use efficient models and rich language. Relying only on counting strategies provides children with inefficient strategies. Counting beans produces bean counters, not mathematicians. For this reason, leaving a child at the concrete level of mastery is not enough and is problematic.

The calculator can be a powerful tool for learning and applying algebra, trigonometry, calculus, probability, and statistics. Children should use calculators when they can demonstrate: 1) *mastery of arithmetic facts*, 2) good *estimation,* and, 3) knowledge of *the concepts.* Without these skills, the calculator becomes an impediment to learning. By allowing calculators in elementary schools, we are producing a generation of innumerates. Schools should therefore ban the use of calculators before the seventh grade. Calculators do not produce mathematicians as typewriters do not produce writers.

We need to keep in mind several essential issues:

(a) Our curriculum must be rigorous, comparable with other high achieving nations,

(b) We must increase and standardize the amount of time devoted to mathematics instruction,

(c) We must establish guidelines for results-oriented math classrooms,

(d) We must furnish principals and supervisors with guidelines and training, enabling them to provide leadership and feedback on mathematics lessons, and

(e) We should vastly increase the number of teachers trained in the identification and treatment of learning disabilities such as dyscalculia.

The process used in developing this Core Curriculum involved reviewing Frameworks from several states, NCTM Standards and Addendum, NCTM Assessment standards, curricula from other nations, and international studies of mathematics achievement, having experiences with children and adults, gathering information from focus groups on mathematics teaching, as well as other resources, and rethinking the existing mathematics curriculum and textbooks.

I hold this vision of the curriculum as desirable for all students and define it as the merging of methodology, content, and assessment. I believe in a curriculum that is:

Accessible

The curriculum should be taught to the greatest majority of children in ways appropriate to their learning abilities, potential, style, and development. The focus should be on whole class teaching augmented with individualization when needed.

Meaningful

The curriculum emphasizes the connectedness to the world beyond the classroom. Connections should be made within mathematics, across disciplines, and to the real world. Students should be shown that the learning of mathematics is a life-long process. Mathematics should be taught and learned as an integral whole and not as isolated facts. For example, fractions should not be taught only during the traditional 3- week period designated for such a lesson but should be revisited throughout the school year in a variety of contexts such as data, probability, and geometry. Problem solving should be the focus of this curriculum. Computation should be explored in a variety of contexts.

Technological

The curriculum incorporates available, appropriate technology as a tool in learning and using mathematics. Children should be given opportunity, where appropriate, to develop and apply their information technology capability in the study of mathematics.

Reflective

The curriculum incorporates experiences, which are exploratory and discovery-oriented, thus enhancing the student's ability to think fluently, critically, and creatively in mathematics. Students should be encouraged and guided to persist and take risks, be actively engaged, and feel confident in their ability to do mathematics. They should be given the opportunity and guidance to construct their own mathematics and then appreciate and acquire the standard universality of mathematics.

Student Focused

The curriculum gives each child the opportunity to learn through meaningful, concrete, useful, and rewarding experiences. This includes understanding linguistic, conceptual, and procedural components of mathematics. This means understanding as well as computational skills are essential for optimal functioning in mathematics. Mathematics should be developmentally appropriate and be based on children's prior knowledge and interest.

Teacher Driven

Although each student constructs his/her schemas of mathematics, teachers create the environment, present the questions to investigate, materials for modeling, and directions for explorations. They assess learning and establish mastery levels. The teacher brings the curriculum to its dynamic state; therefore, the teacher makes it possible by selecting and providing meaningful and productive experiences: explorations, new materials, tasks, and coaching. She appropriately engages in whole-class teaching, small group work, and individual support according to the need of the curriculum, content, and children.

Learning Styles Based

The curriculum should include activities and models, which address the different learning patterns and backgrounds of students. There is a need to progress from the intuitive and concrete level of knowing to the representational, abstract, application, and communication levels. Therefore, the mastery is achieved when a student is able to communicate learning. Students should be given a variety of experiences that allow them to draw upon multiple strategies to solve problems.

Language Sensitive

The curriculum promotes the idea that a teacher of mathematics is a teacher of a language also. Mathematics is a language; it has its own vocabulary, syntax, and rules of translation from English to mathematics and from mathematics to English. This curriculum expects children to express their mathematics understanding clearly through speech and writing and to develop their reading skills as related to mathematics.

Rigorous

The curriculum is designed to adhere to the nature of mathematics—initially it is exploratory, but eventually it is exact, efficient, and elegant. Mathematics is envisioned as broad in scope, including areas beyond computation. Students are expected to be accurate in their presentation of material, and teachers should inculcate effective communication of mathematical ideas and emphasize the rigorous nature of mathematical ideas. Students and teachers must maintain high expectations for themselves and each other. In addition to mastery, mathematics should emphasize meaning of computation and operations. The following principles should inform the content and pedagogy for the curriculum. They can help us design and implement a rigorous curriculum and quality instruction in our classrooms:

1. The goal of mathematics curriculum and instruction in our schools should be to help children acquire the ***mathematical way of thinking*** so that children can recognize and appreciate the applications and beauty of mathematics. The mathematical way of thinking means acquiring the *language*, *concepts, methods*, and *key procedures* and strategies of mathematics.
2. Each mathematical idea consists of three components: *linguistic*, *conceptual*, and *procedural*. ***Mathematics is a language***. It has its own vocabulary, syntax, and rules of communication. Learning mathematics is like learning a second language. We cannot receive, comprehend, and communicate a concept if we do not have a *language container* for the concept.
3. Each mathematics concept passes through ***six levels of knowing***: *intuitive, concrete, pictorial, abstract, applications*, and *communications.*
4. There are key ***prerequisite skills*** for mathematics learning: sequencing, spatial orientation and space organization, pattern recognition (identifying, extending, applying, and creating), visualization, estimation (quantitative and qualitative), deductive and inductive reasoning. The absence or lack of mastery in these pre-requisite skills may manifest in a child having difficulty in acquiring mathematical concepts, skills and/or procedures.
5. Mathematics is the study of ***patterns***—recognizing, extending, applying and creating patterns using quantity and space. *Arithmetic* is the study of patterns in quantity and number relationships. *Geometry* is the study of patterns in shapes and their relationships. *Algebra* is the study of patterns in quantity, space, and variability. In this sense algebra is generalized arithmetic. *Probability* is the study of patterns in chance and its predictability. *Statistics* is the study of patterns in data and its presentations. *Calculus* is the study of patterns in rate of change and their relationships. The aim of mathematics instruction is therefore to help children become proficient in

a) identifying and discerning patterns;
b) extending patterns;
c) applying patterns, and;
d) creating patterns in numbers, shapes, and change.

6. Mathematics is ***the integration of quantity and space***. We identify some aspect or attribute of the space (our environment). We quantify that aspect of space by applying some measurement tools and methods to that spatial aspect or attribute. This results in quantity (numbers), and then we operate on those quantities. These operations provide us some inductive truths. We abstract these inductive truths into abstractions and express them symbolically. This process is called *mathematization.*
7. There are ***key milestones*** in the development of a child's mathematical thinking. These are: number conceptualization, place value, fractions, integers, sense of variability, and spatial sense. The key objective of the school curriculum is to help children develop and master these milestones according to a socially acceptable schedule and in a form using the principles described above. When these milestones are not acquired and internalized by children at the appropriate time, they may experience difficulty in mathematics learning and may manifest learning disabilities.
8. Each of us has a unique ***mathematics learning personality*** falling on a continuum – from quantitative (procedural thinking) to qualitative (thinking in patterns). We process mathematics information according to our mathematics learning personalities. Mathematics instruction should reflect the roles of these learning personalities.
9. A child's mathematics thinking and achievement is a function of the child's cognition and the factors described above. Therefore, in terms of a child's cognition, mathematical thinking can be improved by:
   (a) asking a great deal of real and hypothetical questions,
   (b) using appropriate concrete materials, and
   (c) improving a child's meta-cognition.
10. Effective teaching and optimal learning of mathematics is the process of connections between the content and the learner. For this to happen, the teacher/ tutor must use appropriate instructional strategies and models. These mathematical conceptual models should be universal in nature. An instructional model is universal and effective when it is: *exact, efficient, and elegant.*
11. From this perspective, the focus of mathematics instruction:
    - from Kindergarten through fourth grade should be on mastering number and spatial sense and operations on whole numbers in multiple contexts;

- from the third through fifth grades, the emphasis should be on mastering number relationships, part-to-whole relationships—fractions and probability; comparison of a quantity with a standard—decimals and percents, comparison of two quantities—ratio, rate, and speed, comparison of comparisons—proportion, and variations of different kinds), and integration of quantity and space (applications);
- the work in fifth and sixth grades should be used for reinforcing multiple forms of fractions (rational numbers), developing a clear sense of integers and operations on them, and applications of these skills;
- in seventh and eighth grades, the focus should be on understanding variability and operations on variables and understanding and applying algebraic and geometric models in multiple contexts;
- in high school, the students should master algebraic, geometric, discrete (probabilistic) models to learn coordinate geometry, trigonometry, discrete and continuous models, calculus and their applications to natural, physical and social sciences.

12. Applications at all levels from Kindergarten through 8^{th} grade (and beyond) should be in the context of: intra-mathematical, interdisciplinary, and extracurricular.
13. Every mathematics concept or procedure must begin with a special case, but, fairly quickly, should be taken to its general form (algebraic expression). For example, one may begin with 3 + 5 = 5 + 3, and takes it to glob + slob = slob + glob, $\Delta + \square = \square + \Delta$, and A + B = B + A.
14. It is better to do well in key concepts and procedures (mentioned above), then to do less well in many procedures.
15. Children do experience learning difficulties, disabilities, and problems. They are manifested in the form of (a) developmental, (b) carryover, (c) mathematics anxiety, and (d) dyscalculia. Dyscalculia is to mathematics as dyslexia is to language. A child's difficulty in reception, comprehension and/or production of quantitative or spatial information is called dyscalculia. It is manifested in a child having difficulty in number conceptualization, mastering number relationships, or numerical procedures.
16. **Children rise to higher expectations in positive and conducive learning environments.** Assessment in mathematics should take a variety of forms and be multi-faceted to: monitor student progress, enhance learning, and improve instruction. The learning goals in mathematics, at each grade level, include:
    - Content standards: skill and content objectives that define what every student must know and be able to do, by grade level and course;

- Performance standards: students' work should reflect different levels of knowing (intuitive, concrete, pictorial, abstract, applications, communication) and mastery of required procedures, and
- Student products: tests, tasks, and projects students must successfully complete as a condition of promotion, course completion, and graduation, to demonstrate their skills and understandings. Students with unique needs will use strategies that are considerate of their disability. Bilingual students will use materials appropriate to their needs and ability. For the small number of students who may need the provision, material may be selected from earlier or later stages where this is necessary to enable individual children to progress and demonstrate achievement. Such material should be presented in contexts suitable to the child's age.

KEY CONCEPTS, LANGUAGE, AND PROCEDURES TO BE MASTERED

PREKINDERGARTEN and KINDERGARTEN:

The key objective of the prekindergarten is to provide children quantitative and spatial experiences and terminology in the context of daily language and concrete experiences such as playing games and interaction through toys and manipulatives.

The key objective at the Kindergarten level is to understand number. Number conceptualization is dependent on three key sub-skills: 1-1 correspondence, sequencing, and visual clustering. Children who have not integrated and mastered these key sub-skills will continue to count on fingers and have difficulty in learning arithmetic facts.

1. Numbers at least up to 100
2. Place value (two digit numbers)
3. Counting forward and backward from a given number
4. Identifying simple shapes
5. Measurement using body parts
6. Developing prerequisite skills for mathematics learning
7. Learning and using mathematics language

GRADE ONE:

1. Reading, writing, and expressing numbers in hundreds
2. Understanding place value (three digit numbers)
3. Counting forward and backward by 1’s, 2’s, 10’s from a given number

4. Automatizing addition and subtraction facts (10 x 10 grids)
5. Recognizing, describing, and drawing all the basic figures
6. Multiplication tables of 1, 2, 5, and 10 (as skip counting)
7. Measurement using body parts and go-between units
8. Recognition of fractions (whole, halves, fourths)
9. Developing prerequisite skills for mathematics learning
10. Expanding the language of mathematics and using mathematics language

GRADE TWO:

1. Reading, writing, and expressing numbers in at least two cycles (100,000)
2. Place value any digit number (standard, semi-expanded, and expanded forms), and its application to money and measurement
3. Counting forward and backward by 1's, 2's, 5's, and 10's from any given number
4. Operations on numbers (using mastery of addition, subtraction facts) in these operations with and without borrowing, and application to time, money, perimeter, and measurement
5. Recognizing, describing, and drawing all the basic figures and their relationships
6. Measurement using standard units (whole and halves); calculating perimeters, representation of data
7. Automatization of multiplication tables of 1, 2, 5, and 10
8. Recognition of fractions (1/2, 1/3, ¼, 1/10)
9. Developing, reinforcing, and extending prerequisite skills for mathematics learning
10. Fluent use of the language of above concepts and procedures

GRADE THREE:

1. Reading, writing, and expressing numbers in any number of cycles
2. Place value any digit number (standard, semi-expanded, and expanded forms), and writing money and measurements
3. Counting forward and backward by 1's, 2's, 5's, 10's, 100's and 1/2 from any given number
4. Operations on numbers (addition and subtraction), with and without borrowing
5. Recognizing, describing, and drawing all the basic figures and their relationships

6. Measurement using standard units (whole and halves); calculating perimeters of any shape and area of rectangles, squares, and shapes made up of rectangles and squares
7. Automatization of tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12; relate and automatize multiplication and division facts; properties of numbers
8. Simple fractions, their relationships, and adding fractions with the same denominators
9. Representation of data
10. Developing, reinforcing, and extending prerequisite skills for mathematics learning
11. Fluent use of the language of above concepts and procedures

GRADE FOUR:

1. Reading, writing, and expressing numbers in any number of cycles and simple decimal numbers
2. Place value any digit number (standard, semi-expanded, and expanded forms)
3. Counting forward and backward by 2’s, 5’s, 10’s, 1/2, 1/3, 1/4, 1/4, 1/10 from any given number
4. Operations on multi-digit numbers (addition, subtraction, multiplication, and division), with and without borrowing
5. Recognizing, describing, and drawing all the basic figures and their relationships
6. Measurement using standard units (whole and halves)
7. Automatization of tables 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12; relate and automatize multiplication and division facts with properties (communicative, association, and distribution); properties of numbers
8. Simple fractions and operations of addition and subtraction on simple fractions
9. Representation of data
10. Developing, reinforcing, and extending prerequisite skills for mathematics learning
12. Fluent use of the language of above concepts and procedures

GRADE FIVE:

1. Reading, writing, and expressing numbers in any number of cycles and decimal numbers

2. Place value any digit number (standard, semi-expanded, and expanded forms)
3. Operations on numbers (addition, subtraction, multiplication, and division), with and without borrowing
4. Counting forward and backward by 2's, 5's, 10's, 1/2, 1/3, 1/4, 1/4, 1/10 or .1 from any given number
5. Properties of numbers and divisibility tests for 2, 3, 4, 5, 6, 9, and 10
6. Recognizing, describing, and drawing all the basic figures and their relationships; calculate perimeter of any polygon and area of rectangles, triangles, and derived shapes made of these shapes; genealogy of quadrilaterals
7. Measurement using standard units (correct up to tenths)
8. Relate fractions, decimals, percents, and ratio; operations on
9. Representations of data and making inferences
10. Developing, reinforcing, and extending prerequisite skills for mathematics learning
11. Fluent use of the language of these concepts and procedures

Milestones: Fluency With Whole Numbers

1) By the end of Grade 3, students should be proficient with the addition and subtraction of whole numbers.
2) By the end of Grade 5, students should be proficient with multiplication and division of whole numbers.

GRADE SIX:

1. Reading, writing, and expressing numbers in any number of cycles and decimal numbers
2. Place value any digit number including decimals (standard, semi-expanded, and expanded forms – including exponential form)
3. Operations on fractions and integers (addition, subtraction, multiplication, and division), in different forms
4. Counting forward and backward by any number (whole, fraction, decimal) from any given number
5. Divisibility rules in general and properties of numbers
6. Recognizing, describing, and drawing all the basic figures and their relationships; calculate perimeter of any polygon and circle and area of rectangles, triangles, and derived shapes made of these shapes and circles; genealogy of quadrilaterals and triangles

7. Measurement using standard units (correct up to tenths)
8. Relate fractions, decimals, percents, ratio, proportion, and variations—linear and simple non-linear involving equations;
9. Representations of data and making inferences
10. Developing, reinforcing, and extending prerequisite skills for mathematics learning
11. Fluent use of the language of these concepts and procedures

GRADE SEVEN:

1. Place value any digit number including decimals (standard, semi-expanded, and expanded, and exponential – including negative exponential forms)
2. Genealogy of real numbers; operations (addition, subtraction, multiplication, and division) on real numbers; law of exponents,
3. Recognizing, describing, and drawing all the two and three dimensional shapes and their relationships; calculate areas of two dimensional figures, surface areas, total areas, and volumes of solids
4. Relate fractions, decimals, percents, ratio, proportion, exponents, and variations
5. Number theory: properties of numbers
6. Representations of data and making inferences
7. Rules of exponents
8. Integration of geometry and algebra: Representing and solving simple linear and non-linear equations
9. Fluent use of the language of above concepts and procedures

Milestones: Fluency With Fractions

1) By the end of Grade 4, students should be able to identify and represent fractions and decimals, and compare them on a number line or with other common representations of fractions and decimals.
2) By the end of Grade 5, students should be proficient with comparing fractions and decimals and common percents, and with the addition and subtraction of fractions and decimals.
3) By the end of Grade 6, students should be proficient with multiplication and division of fractions and decimals.
4) By the end of Grade 6, students should be proficient with all operations involving positive and negative integers.

5) By the end of Grade 7, students should be proficient with all operations involving positive and negative fractions.
6) By the end of Grade 7, students should be able to solve problems involving percent, ratio, and rate and extend this work to proportionality.

Milestones: Geometry and Measurement

1) By the end of Grade 5, students should be able to solve problems involving perimeter and area of triangles and all quadrilaterals having at least one pair of parallel sides (i.e., trapezoids).
2) By the end of Grade 6, students should be able to analyze the properties of two-dimensional shapes and solve problems involving perimeter and area, and analyze the properties of three-dimensional shapes and solve problems involving surface area and volume.
3) By the end of Grade 7, students should be familiar with the relationship between similar triangles and the concept of the slope of a line.

GRADE EIGHT:
A comprehensive course in algebra that integrates algebraic, probabilistic, and geometric models, or Math I for majority of students.
GRADE NINE:
Comprehensive look at geometry, or Math II
GRADE TEN:
Integration of algebra and geometry, coordinate geometry, trigonometry, functions, and discrete models, or Math III
GRADE ELEVEN:
Comprehensive look at functions, limits, and rate of change, or Math IV
GRADE TWELVE:
Calculus and discrete models, or Math V

ROUTINE ELEMENTS OF A
MATHEMATICS LESSON

1. **Counting**
2. **Oral Facts**
3. **Estimation**
4. **Prerequisite Skills**

5. Concept Building
6. Recording
7. Number Stories
8. Review

Counting

- **Number grid**
- **Forward**
- **Backward**
- **Skip Counting**
- **By Fractions**
- **By Decimals**

  (3 to 5 minutes each day)

COMPONENTS OF A MATHEMATICS LESSON
(Routines and Structures for Delivery of Mathematics Instruction)

I. MENTAL ARITHMETIC/MATHEMATICS (10 minutes) [Entire class]

A. Counting (forward, backward, skip counting by whole numbers, fractions, and decimals)
B. Counting using number grid
C. Oral arithmetic facts and extending facts using larger numbers and distributive property
D. Oral mathematics to show patterns, relationships and forming conjectures)
   1. Problem of the day (individual student)
   2. Discussion of the problem (small groups)

II. MAKING CONNECTIONS: LINKING CONCEPTS AND PROCEDURES (10 minutes)

A. Review of materials and summary of previous day's work (comments and dealing with homework) **[Entire class]**
B. Multiple assessments
C. Extensions and applications

III. PREREQUISITE SKILLS (Embedded in the lesson)

A. Sequencing
B. Spatial orientation/space organization

C. Pattern recognition, extensions, creating numerical and geometrical patterns
D. Visualization
E. Deductive reasoning
F. Inductive reasoning

IV. INTRODUCING A NEW LANGUAGE, CONCEPT, AND/OR PROCEDURE (30 minutes)

A. Levels of Knowing:

1. **Intuitive** (making connections with previous learning; Socratic questioning)
2. **Concrete** (conceptual and model development; Models—exact, efficient, and elegant)
3. **Representational** (pictorial, graphical, graphing calculator, compugraphics, recording and pre-symbolic)
4. **Abstract** (symbolic work on paper, calculator or computer; use of formal language and symbols, tests, quizzes and examinations)
5. **Applications** (intra-mathematical, interdisciplinary, extracurricular, word problems, projects, modeling, simulations)
6. **Communications** (demonstration of mastery and competence through written, graphical, compugraphic, concrete, tests, examinations, peer-teaching, designing tests and problems, or oral mathematics.)

B. Models:

1. Discrete (quantitative)
2. Continuous (qualitative)
3. Integrated
4. Universal

C. **Mathematics as a Second Language** (Integrated in the lesson)

1. Vocabulary and terminology
2. Structure of mathematical language (syntax)
3. Translation from Math to English (Writing number stories, making conjectures, definitions, articulating patterns)
4. Translation from English to Mathematics (Solving word problems) **[Socratic questioning and key questions]**

5. Review (Consolidation of learned material) (5 minutes)
6. Summary and review at the end of the class
7. Summary at the beginning of the next class

8. Repetition, rehearsal, transfer

D. **Homework** (5 minutes)
1. Cumulative (1/3)
2. Representation of classroom work (1/3)
3. Preview and number stories (1/3)
4. Challenge (1problem)

A lesson should be at least one hour long. Mathematics should be taught each day of the week. Each lesson's purpose is to consolidate previous learning; introducing new concepts, making connections, and helping students become independent and proficient learners. Most of the lesson should be a whole-class activity, except when it is a tutorial. Assessment should be embedded in the lesson with periodic formal assessments. When a collection of topics, concepts and procedures is done, there should be a written assessment. No written assessment should be done before some oral assessment or group assessment is done.

FRACTIONS: AN INTRODUCTORY LESSON

I. Grade level: 1st through 10th

II. Goals for Students:

A. This lesson will help:
- Develop the concept of fractions,
- Learn key vocabulary and language containers for fractions,
- Understand and determine how fractions are used in our daily lives,
- Discover different forms of writing fractions,
- Create fractions using paper strip fraction model—whole, half, third, fourth, sixth, eighth, and twelfth fractional values,
- Understand 1 in different fractions, and
- Represent fractions as numbers on a number line.

B. Materials:

Chalkboard, chart paper, colored paper strips, which have been prepared in advance to save time, markers, newspapers, pencil and paper, poster paper, scissors

III. For the Teacher to Consider:

You want students to know that:

A. Fractions are manifested as:

- *Part to whole relationship* (both in discrete and continuous models)

  Examples: (1) The boys constitute ⅔ of the class (discrete). (2) This slice is ⅛ of the pizza (continuous).
- *Comparison of quantities*: comparison of any two quantities (Ratio; Example: What fraction of the class are boys? What is the ratio of boys to girls?);
- *Comparison of one quantity with a standard quantity* (Example: decimal number and percent are comparisons of a given quantity with a standard such as 10, 100, 1000, etc.); comparison of comparisons (Proportions are comparisons of ratios);
- *Applications*: Probability (There is ⅓ chance that I will hit the ball. His average is 313/1000.)

B. Understanding fractions mean:

1. The whole is being divided;
2. There is a certain number of parts;
3. All the parts are equal; and
4. All the parts together make the whole.

C. Fraction, a process and a number

1. Find ⅔ of the weight of this package.
2. Locate the number ¾ on the number line.

IV. Key Vocabulary Words:

Students need to know the meanings, examples, applications and use of these terms and definitions:

Fracture, fraction, fractal, whole, part of, out of, cut, comparison of numbers, separate, equal, equivalent, break into parts, divide, a piece of, section, less than a whole, more than a whole, numerator, denominator, half, third, fourth, ratio, proportion, sector, section, sixth, eighth, twelfth

V. "How to" of the lesson:

A. Intuitive level of knowing:

At this level, you want to know what the students already know about fractions and related concepts. The Socratic method of questioning is best suited for this activity. One should ask such questions as: "What does a fraction mean to you?" "Where do you use fractions?" "Can you give me an example of a fraction?" or "Where do you see fractions?" etc.

1. Students will discuss and record what a fraction means to them. They will explore real life examples, where they experience and deal with fractions.

Each student will give one example of a fraction from his/her life. They will cite examples of fractions in their everyday life.

2. Making change
3. Telling time
4. Dividing things
5. Sharing things
6. A chart will be created demonstrating how fractions are used.
7. Students will learn fractions, and will write them in several forms.
8. **The homework assignment** will be to find examples of fractions from newspapers, periodicals, books, daily activities, etc. The objective of the homework is to (a) continue the learning from classroom to home (b) reinforce what has been learned in the classroom, and (c) and extend learning for students from classroom to applications. Therefore, the homework assignment should have three components:
    - cumulative— representative of problems from previous concepts and procedures,
    - problems that look exactly like the ones you have done in class today, and
    - further exploration to create in-depth understanding and or apply the concept.

B. Concrete level of knowing:

Every new concept should be introduced through some concrete activity or referred to an already used concrete activity. Concrete activities provide students with the opportunity to participate in an activity that will form conceptual models. When students have worked with appropriate and efficient concrete models, they can abstract ideas from them. In this concrete activity, students will work in pairs using colored paper strips, scissors, and pencils.

Talk about machines they encounter in their lives and then introduce the fraction machine idea to students. At this level, you want students to construct, make, or design something to understand the concept at the concrete level. In order to integrate all the conceptual components of the concept of a fraction, students must construct fractions at least once. Already made fractions are not good enough. After making fractions once, you can use already made fraction bars. The strip will be introduced and will be named a unit. Students will be asked to identify one whole, two wholes, etc.

Questions to be asked and language to be used:

Question: Can you show me a whole?

Question: Can you show me two wholes? Etc.

Students write one whole on one side of the fraction strip and "1" on the other side. After appropriate answers to these questions, then ask:

Question: When the whole is put through the half machine, what happens?
Answer: Two halves come out.

The machine idea works as a metaphor: the change machine gives change; the bread machine gives bread; and the half machine gives halves.

1. **Students will create fraction strips in whole, half, third, fourth, sixth, eighth, and twelfth values.**

Students will strategize in small groups and in a whole group, on proper folding techniques to make accurate fraction equivalencies.

Questions to be asked and language to be used:

- Put the whole through the half machine.
- What will you get? How many pieces will you get?
- What is the name of each piece?
- Write ½ on one side of each piece and 1 out of 2 on the other side of each piece.
- Is ½ smaller than one whole?
- Is ½ bigger than one whole?
- How many halves are in a whole? Write $1 = \frac{2}{2}$ on the chalkboard.
- How many halves are there in two wholes? Three wholes? Four wholes? Ten wholes? Hundred wholes? Record all of these on the chalkboard.

2. **Challenge**:

- How many halves are there in A wholes, B wholes?
- Hold five wholes in your hands and ask: Read these in two ways.
- Student should answer: Five wholes or 10 halves ($\frac{10}{2}$). Record this on the chalkboard.
- Repeat this with 1½, 2½, 3, etc.

C. Pictorial representation:

As the students construct different fractions, you should draw them on the board and then ask students to draw fractions. Given different types of shapes, ask them to identify different parts of the whole and ask them to shade the appropriate fractions of the whole. Giving wholes and parts shaded, ask them to identify the appropriate fractions. The same process and questions that were used at the concrete level should be used at the pictorial and representational levels. The objective of instruction at this level is to take children from the concrete level to a representation of the idea of part-to-whole. This process leads a child to the development of language and concept of fractions that is not tied to objects.

D. Abstract level of knowing

The real understanding of a concept is not evident until it is represented abstractly and symbolically. As all formal communication of mathematics ideas and assessment of its mastery need to be demonstrated symbolically, it is important that children transcend the concrete and pictorial representations of mathematics ideas into symbolic abstract forms. Abstractions begin with labeling. Therefore, the first step is for students to label fraction strips in several ways: (a) Numbers, (b) Word, (c) Fractional parts, and (d) Equivalent values.

The naming information will be recorded on strips (Example: ½ (one-half), 1 out of 2; ⅓ (one-third), 1 out of 3; ¼ (one-fourth), 1 out of 4, ½ of ½; ⅙, 1 out of 6, ½ of ⅓, ⅓ of ½; ⅛, 1 out of 8, ½ of ¼, ½ of ¼, ½ of ½ of ½; etc. The symbols such as: ½, ⅓, ¼, etc. indicate standard number names of these fractions, whereas terms such as 1 out of 3, 1 out of 8, etc. represent linguistic and conceptual representations of fractions. Symbolic expressions such as ½ of ⅓, ⅓ of ¼, represent linguistic, procedural, and conceptual forms of fractions.

The teacher should repeat the questions from concrete and pictorial sections applying them to these new fractions. Students will be asked to show different fractional values.

Questions to be asked and language to be used:

- Hold two whole strips in your hand and ask: What is the name of these together in wholes, in halves, in thirds, in fourths? etc.
- Hold a whole and half in your hand and ask: What are different ways of naming this? In halves? In thirds? In fourths?

Questions to be asked:

Question: Give different names of the piece (hold ¼ in your hand).
Answer: 1/4; a quarter; 1 out of 4; ½ of ½.

Question: Give different names of this piece (hold ⅛ in your hand).

Answer: ⅛; ½ of ½ of ½; ½ of ¼; ¼ of ½; 1 out of 4; one eighth.
Question: What if I had 1/12 in my hand, what would be the different names of this fraction?

Answer: One-twelfth; ½ of ⅙; ⅙ of ½; ¼ of ⅓; ⅓ of ¼; ⅓ of ½ of ½; ½ of ½ of ⅓; 1/12.

Students will create true and false statements about particular fractions. Each time you introduce a new fraction, you ask them (in groups of two or three or alone) to write five true statements and five false statements. One group of students reads a statement and another identifies the statement as true or false. Then the group proves it true or false by using the fraction strips. If the statement is false, they convert it into a true statement. If the statement is true, ask them to convert it into a false statement using their fraction to demonstrate their statement. They also use the statement in more than one way.

These different ways of naming fractions will be recorded on the chalkboard and by students on their papers. This kind of symbolic representation will introduce the idea of equivalent fractions.

Take one of the statements presented by one group and ask each group to create a story using these particular fractions. Stories will be shared with the class**.**

VI. Understanding a Unit as a Fraction in a Different Forms:

Before students can perform operations on fractions, it is important for them to express a unit in terms of a fraction in different forms. In this section, students will discover different fractional values of a unit (1 whole) and will be able to express an equivalent fraction.

Questions to be asked and language to be used:

The teacher should point to the collection of fractions both in the concrete and pictorial forms and ask students to read the whole as a fraction.

Question: Give a fraction name for the unit strip.

Student answers: 2/2

Question: Give another (different) fraction name of a whole.

Students answer by saying 3/3.

Question: Any other name?

You need to get them to realize that 1 = 2/2 = 3/3 = 4/4 = 5/5 = 6/6 = 8/8 =... Now take a whole in your hand and ask a student:

Question: What is the name of this?

Student should say: One whole**.**

Question: What if I put it through the fifteenth machine, how many pieces will I get?

Student: Fifteen pieces.

Question: What is the name of each piece?

Student: One-fifteenth.

Question: What else can you call it?

Student: One out of fifteen.

Questions: Any other name?

Student: 1/3 of 1/5.
Question: How many fifteenths will make a whole?
Student: Fifteen.
Question: How do you write one whole as fifteenths?
Student: 15/15.

Record this on the chalkboard: **1 = 2/2 = 3/3= 4/4= 5/5 = 6/6 = 8/8 = 15/15 =**
Ask several questions to ascertain different names for the piece. Repeat this with every child in the class by changing the name for the machine (sixteenth, twentieth, hundredth, eleventh, $\blacktriangle^{th}$ machine, A^{th} machine, B^{th} machine, $(A + B)^{th}$ machine, etc. Each one of these machine questions should be addressed to a different student.

The selection of the machine question must be appropriate to the level of the student. Sometimes, when you find that a student is having difficulty, you may go back to the fraction display such as two halves and three thirds or even half, or 1/3-machine. You may also like to go back to a concrete example if the student has not understood the concept clearly.

The objective is that the student should know 1 in many fractional forms. When students can give you 1 as a fraction, they are prepared for operation on fractions. The sequence of questions such as the following can help in the development of this idea.

E. Communications level of knowing

One knows something when he/she can communicate that knowledge. You want students to develop the ability to communicate what they know; the following sequence of questions can help:

Question: Can anyone see any pattern in the fractions we used for 1?
Answer: Yes! The numerator was always equal to the denominator.
Question: Can somebody make a conjecture about this pattern?

Your objective is to help students to realize that a fraction is equal to 1 when the numerator is equal to the denominator. When this happens you identify the student who saw the pattern and name this conjecture after his/her name. This person will be the expert on 1 in fraction form. When a student experiences any difficulty in identifying a fraction as one or expressing one as a fraction, this person will be consulted. This is a good example of inductive reasoning and extension of a pattern observed.

Question: Please write five fraction names for 1 on your paper**.**
Question: Can somebody give me a fraction that is equal to 1 without using any numbers in it?

You want to get answers such as: M/M. mango/mango, etc. If no student gets that form, help him or her get it by referring to the conjecture.

Question: Can somebody give me a fraction that is equal to 1 and uses numbers and letters?

You want to get fractions such as: (M + 10)/ (M + 10). Using the same questioning technique, extend this idea to recognize fractions that are smaller and bigger than one. You need to end up with a conjecture identifying fractions less than 1 and greater than 1. This technique is a good example of extending patterns and inductive and deductive reasoning.

Students will:

1. Brainstorm to find out what fractions mean to them, examples of how fractions are used and fractions on a board using several forms
2. Construct fractions using different shapes and write statements using fractions
3. Compile a variety of written samples of fractions from media
4. Maintain a fraction vocabulary list in a math notebook or a display board
5. Show fractional values on a number line
6. Make true and false statements about fractions
7. Write fraction stories using model as source

IX. Assessment:

Assessment will be ongoing as fraction concepts are developed and expanded in lessons. Assessment will rely a great deal on oral and written work.

X. Homework Activities:

Students will compile fraction sample data from newspapers and periodicals. A collage will be created using chart paper. Students will also write fraction stories as a homework assignment. The teacher will assign appropriate homework.

Center for Teaching/Learning of Mathematics

Programs and Services

CT/LM offers programs and materials to assist teachers and parents to help children and adults with learning difficulties in mathematics. We conduct regular **workshops, seminars,** and **lectures** on topics such as:

1. How does one learn mathematics? This workshop focuses on psychology and processes of learning mathematics. Participants study the role of factors such as: Cognitive development, language, mathematics learning personality, pre-requisite skills, and conceptual models on learning mathematics—concepts, skills, and procedures. They learn how children achieve the key mathematics milestones: number, arithmetic facts, place value, fractions, integers, algebraic thinking, and spatial sense. They learn strategies to teach their students these concepts more effectively.

2. What are the causes of learning problems in mathematics? This workshop focuses on understanding the nature and causes of learning problems in mathematics. Participants become familiar with diagnostic instruments and remedial approaches for dealing with learning problems in mathematics. They learn strategies for working with children and adults with learning problems and disabilities in mathematics, such as dyscalculia, more effectively.

3. Content workshops. These workshops are for teachers and parents on teaching key mathematics concepts and procedures. Examples: **How to teach arithmetic facts easily? How to teach fractions with better understanding and results? How to take students from arithmetic to algebra easily?** In these workshops, we use a new approach called **Vertical Acceleration,** where we begin with a very simple concept from arithmetic and take it to the algebraic level.

At the Center, we offer **individual diagnosis** and **tutoring services** for children and adults to help them with their mathematics learning difficulties and learning problems, in general, and dyscalculia, in particular. We also provide:

1. Consultation with and training for parents and teachers to help their children cope with and overcome their anxieties and difficulties in learning mathematics.

2. Consultation services to schools to help evaluate their mathematics programs and design new programs or supplement existing ones in order to minimize the incidence of learning problems in mathematics.

3. Assistance for the **adult student** who is returning to college and has anxiety about his/her mathematics.
4. Assistance in test preparation **(SSAT, SAT, GRE, MCAS**, etc.)

About the Author

Professor Mahesh Sharma is the former President and Professor of Mathematics Education at Cambridge College. He is the founder and President of the Center for Teaching/Learning of Mathematics, Inc. of Wellesley and Framingham, Massachusetts. *Berkshire Mathematics* in Reading, England, was founded to facilitate his work in the UK and Europe. Internationally known for his groundbreaking work in mathematics education, he is an author, teacher and teacher-trainer, researcher, consultant to public and private schools, as well as a public lecturer. He has been the Chief Editor and Publisher of *Focus on Learning Problems in Mathematic*s, an international, interdisciplinary research mathematics journal with readership in more than 90 countries, and the Editor of *The Math Notebook*, a practical source of information for parents and teachers devoted to improving teaching and learning of mathematics for all children. He provides direct services of evaluation and tutoring for students (children as well as adults) who have learning disabilities such as dyscalculia or face difficulties in learning mathematics. Professor Sharma works with teachers and school administrators to design strategies to improve mathematics curriculum and instruction for all.

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